OPEN-ENDED QUESTIONS FOR MATHEMATICS

Appalachian Rural Systemic Initiative
PO Box 1049
200 East Vine St., Ste. 420
Lexington, KY 40588-1049
http://www.arsi.org/

DEVELOPED BY DR. RON PELFREY, MATHEMATICS CONSULTANT

AND

PROVIDED AS A SERVICE OF THE ARSI RESOURCE COLLABORATIVE UNIVERSITY OF KENTUCKY

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This packet contains open-ended questions for grades 4, 5, and 8 as well as open-response questions for Algebra I / Probability / Statistics and Geometry. The questions were developed with two separate intentions.

Before stating these intentions, let's examine the differences – as used in this packet – between "open-ended" and "open-response." In this set of materials, open-ended refers to a question or problem which has more than one correct answer and more than one strategy to obtain this answer. Open-response refers to a question or problem that may only have one correct answer or one strategy to obtain the answer. In both open-ended and open-response mathematics problems, students are expected to explain or justify their answers and/or strategies.

Now for the intentions for the use of these questions. The questions identified for grades 4, 5, and 8 should be used as classroom practice questions. Students can either work with them as members of cooperative groups or the teacher can use the questions for demonstration purposes to illustrate proper use of problem solving strategies to solve problems – as practice either for CATS or for other problem solving situations that students may encounter. The problems are not intended to be ones that can be solved quickly or without thought. However, the challenge provided by these questions should elicit classroom discussion about strategies that may or may not be obvious to the average student. Each of the questions is correlated to the Core Content for Assessment for Grade 5 (the grade 4 and grade 5 questions) or for Grade 8 (the grade 8 questions). If a teacher receiving a copy of these questions does not have the Core Content for Assessment coding page, she/he may contact either the ARSI Teacher Partner in his/her district, the ARSI office (888-257-4836), or the ARSI website of the University of Kentucky resource collaborative at http://www.uky.edu/OtherOrgs/ARSI/curriculum.html then click on *Assessments*.

The high school questions were developed as part of professional development provided to mathematics teachers on how to adapt textbook or other problem sources into open-ended questions. As presently configured, many of these questions can be used in classrooms for assessment purposes. However, the teacher should consider modifying the problems to provide additional practice to their students on how to answer open-ended questions. Assistance in helping teachers in this modification can be found on the Kentucky Department of Education website at http://www.kde.state.ky.us/ or through professional development provided by ARSI or the Regional Service Center support staff in mathematics.

If you have any questions about the use of these materials, please contact the ARSI Resource Collaborative at the University of Kentucky (888-257-4836).

GEOMETRY OPEN-RESPONSE QUESTIONS

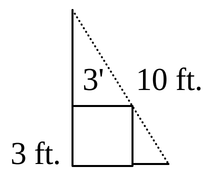
To the Teacher:

This set of questions is intended to be used with students for practice, i.e., questions can be given for students to work with in small groups or for homework. After completion of individual questions, using the answering model that has been established in the classroom, each solution and solution strategy should be discussed among members of the entire class. Many of the questions are more difficult geometry questions than students will see on the KCCT, but the strategies they learn in answering these questions should be similar.

It is not intended that the questions be given as a block, but that there be ample opportunities for discussion between questions. It is not necessary that all questions be asked or that they are asked in any particular order. It should be your decision as to which questions to use with your students.

The solution key includes a cross-reference to the Core Content for Assessment descriptor that is being addressed for each question. It is suggested that, even if you don't use all questions, you select questions from a variety of different descriptors.

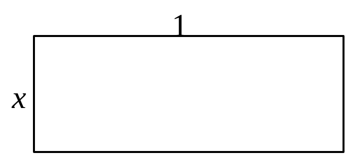
1. A 10-foot ladder leaning against a vertical wall just touches the corner of the box shown. How far up the wall does the ladder reach?



2. Liz said that she would take Jill to lunch if she could find the only two rectangles whose dimensions are integers and whose area and perimeter equal the same number. What should Jill's answer be?

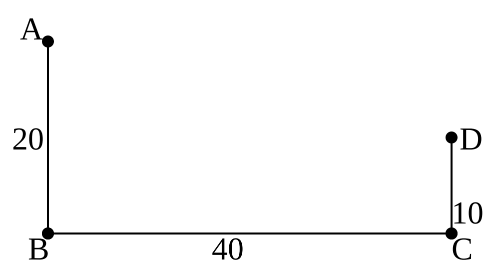
3. What percent of the area of a circle is enclosed by an isosceles triangle one of whose sides is the diameter of the circle?

4. Find x so that a vertical line that cuts the rectangle into two shapes yields (1) a square and (2) a rectangle that is similar to the original one.



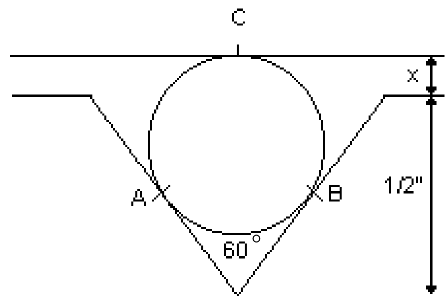
5. Given two equilateral triangles, find a third one whose area is the sum of the other two.

6. Tim must get from point A to point D and must touch some point between B and C along the way. (For example, he could go from A to B to D; or A to the midpoint between B and C and then to D.) What is the shortest length of such a journey satisfying these conditions?

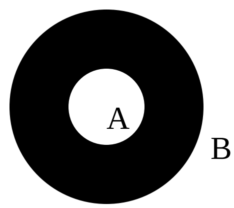


7. The end points of a diameter of a circle are (3, 9) and (11, 3). A triangle inscribed in this circle has two of its vertices at the given points. Find the coordinates of all points at which the third vertex of the triangle can be located for this triangle to have its maximum possible area.

8. If A, B, and C are points of tangency and the diameter of the circle is 3/8 inch, find x.



9. Segment AB, which is one inch long, is tangent to the inner of two concentric circles at A and intersects the outer circle at B. What is the area of the annular region (the ring) between the circles?



10. A running track is made up of two concentric circles. If the track is three meters wide, what is the difference in the two circumferences, i.e., the difference in running on the outside lane versus the inside lane?

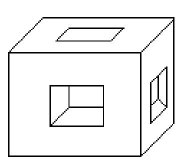
11. A recent advertisement read:

> **Waterfront Property**
> This triangular piece of property has an 895 foot river frontage and a 772 foot road frontage. The remaining 925 foot side is shared with a lumber yard.

What is the acreage of this property? (1 acre = 43,560 sq. ft.) Round to the nearest tenth acre.

12. A ball was floating in a lake when the lake froze. Someone removed it later (without breaking the ice), leaving a hole 24 cm across at the top and 8 cm deep. What was the diameter of the ball?

13. A wooden cube has edges 3 meters in length. Square holes with sides one meter in length and centered in each face are cut through to the opposite face with the edges of the holes parallel to the edges of the cube. What is the entire surface area, including the inside?



14. A semi-elliptical tunnel whose base has a width of 6 meters has to be of such a height that a truck with a height of 4 meters and width of 2 meters will just fit through it. What is the height of the tunnel?

15. Suppose you have a rectangular piece of paper whose dimensions are m x n, where $m>n$. The paper may be rolled to form a cylinder in two simple ways:
    - The edges of length m may be brought together and taped, or
    - The edges of length n may be brought together and taped.

    What is the ratio of the volume of the two cylinders? Which cylinder will have the greatest volume?

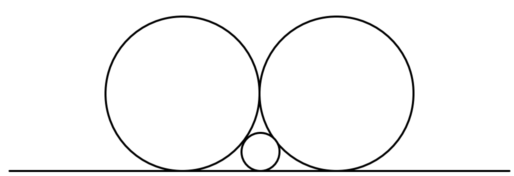
16. A cylindrical can of constant volume has its height increasing at a constant rate of 25% per day. At what rate will the radius be changing?

17. Find the distance between the parallel sides of a regular octagon of side s.

18. A horse is tethered to a rope at one corner of a square corral (outside of the corral) that is 10 feet on each side. The horse can graze at a distance of 18 feet from the corner of the corral where the rope is tied. What is the total grazing area for the horse?

19. Triangle ABC is isosceles with base AC. Points P and Q are respectively on CB and AB such that $AC = AP = PQ = QB$. What is the measure of angle B?

20. Two logs sit side by side so that they are tangent to the ground. Obviously there is enough room between the two logs to place another small log, also tangent to the ground. If the two larger logs are eight inches in radius, what is the radius of the smaller log? Generalize so that if the radius is x for the two larger logs, what is the radius of the smaller log?



GEOMETRY SOLUTIONS

1. *Pythagorean theorem.* (H-2.2.4) $(x + 3)^2 + (y + 3)^2 = 100$

   Similar triangles. (H-2.2.3) $\frac{x}{3} = \frac{3}{y} \Rightarrow y = \frac{9}{x}$

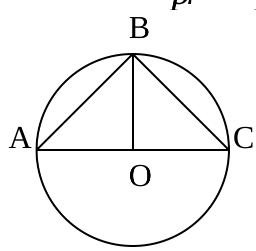
   \ $x^4 + 6x^3 - 82x^2 + 54x + 81 = 0 \Rightarrow x = 5.92$ or 1.52. Then the ladder reaches 8.92 or 4.52 feet up the wall.

2. *Perimeter and area.* (H-2.3.1) $ab = 2a + 2b = 2(a + b)$

a	*b*	*ab*	*2a + 2b*
1	1	1	4
2	2	4	4 + 4 = 8
4	4	16	8 + 8 = 16
6	3	18	12 + 6 = 18

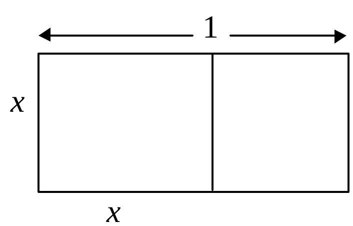
Thus, the only rectangles are a 4 x 4 and a 6 x 3.

3. *Area.* (H-2.3.1) Area of circle 0 is πr^2. Area of ΔABC is $\frac{1}{2}(2r)\,r = r^2$. The percent of the area of the circle is $\frac{r^2}{pr^2} = \frac{1}{p} \approx 0.318 = 31.8\%$.



4. *Proportions including geometric mean.* (H-2.1.4)

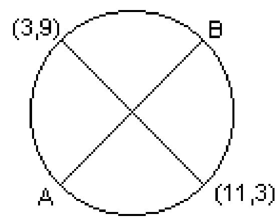
$$\frac{1}{x} = \frac{x}{1-x} \Rightarrow x = \frac{-1+\sqrt{5}}{2} \approx 0.618 \text{ (A golden rectangle.)}$$



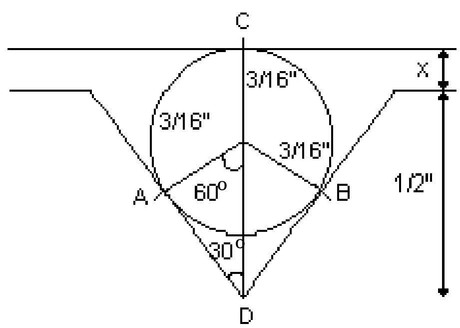
5. *Pythagorean theorem.* (H-2.2.4) The third equilateral triangle can be found by using a side of each given triangle as a leg in a right triangle. The result will be the triangle on the hypotenuse.

6. *Reflections; Pythagorean theorem.* (H-2.1.1, H-2.2.1, H-2.2.4) Reflect point D through line BC down to point D'. Going from point A to the line BC and then to the point D is the same as going from A to BC to D'. Since the shortest distance from A to D' is a straight line, we can use the Pythagorean theorem to find AD' = 50.

7. *Ratio measures such as slope; how algebraic procedures and geometric concepts are related.* (H-2.1.4, H-2.3.2) There are two locations for the third vertex — the points of intersection of the perpendicular bisector of the diameter and the circle, i.e., points A and B on the following diagram. The center of the circle is $\left[\left(\frac{3+11}{2}\right),\left(\frac{9+3}{2}\right)\right]$ = (7,6), the slope of the diameter is $\frac{9-3}{3-11} = -\frac{3}{4}$. So the slope of the perpendicular bisector is $\frac{4}{3}$. The two locations are A = (4, 2) and B = (10, 10).



8. *Use Pythagorean theorem (30/60/90 right triangle); how ratios relate to right triangles.* (H-2.2.4, H-2.1.4) $\frac{1}{16}$"



$\angle$ AOD = 60°, $\angle$ ADO = 30°, CO = 3/16" = AO = BO; DO = 6/16"

9. *Use Pythagorean theorem; how algebraic procedures and geometric concepts are related.* (H-2.2.4, H-2.3.1)
   The center of the circle and points A and B form a right triangle with the radius of the inner circle being the length of one side and the radius of the outer circle the hypotenuse. Let r be the radius of the inner circle and R be the radius of the outer circle. Then $R^2 = r^2 + 1$, or $R^2 - r^2 = 1$. The area of the ring is
   $\pi R^2 - \pi r^2 = \pi\,(R^2 - r^2) = \pi \bullet 1 = \pi$.

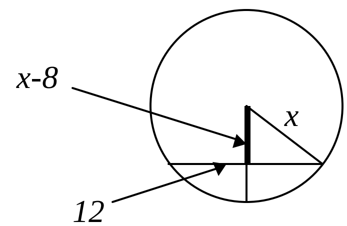
10. *How properties of geometric shapes relate to each other.* (H-2.3.1)
    6π

11. *How algebraic procedures and geometric concepts are related.* (H-2.3.1)

Using Heron's formulas, A = $\sqrt{s(s-a)(s-b)(s-c)}$ where s = $\dfrac{a+b+c}{2}$ = 1296 feet. Then the area is 317,854 square feet. Converting this to acres, we arrive at 7.3 acres.

12. *Using Pythagorean theorem.* (H-2.2.4)
Assuming the ball floated with its center above the water (since it was removed without breaking the ice):

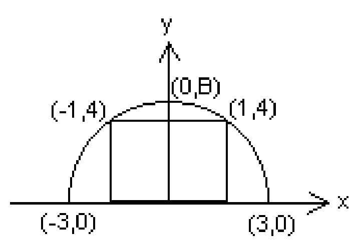
$(x - 8)^2 + 12^2 = x^2$
$x = 13$
\ Diameter = 26 cm



13. *Properties of geometric shapes.* (H-2.2.6)
Each exterior unit square that is removed exposes 4 interior unit squares, so the entire surface area is:
$6 \times 3^2 - 6 \times 1 + 6 \times 4 = 72\ m^2$

14. *How equations, lines, and curves are models of the relationship between two real world quantities; how algebraic procedures and geometric concepts are related; how position in the plane can be represented using rectangular coordinates.* (H-2.3.2)

Using a Cartesian coordinate system, the equation for the ellipse is $\dfrac{x^2}{3^2}+\dfrac{y^2}{B^2}=1$



Substituting the value (1, 4): $\dfrac{1}{9}+\dfrac{16}{B^2}=1 \Rightarrow B = 3\sqrt{2} \approx 4.24\,meters$

15. *How algebraic procedures and geometric concepts are related; construct geometric figures.* (H-2.2.6)

For cylinder 1: Height = m and circumference = n. Since $2pr = n$, $r = \frac{n}{2p}$.

Thus, $v_1 = p\left(\frac{n}{2p}\right)^2 m = \frac{mn^2}{4p}$.

For cylinder 2: Height = n and circumference = m. Since $2pr = m$, $r = \frac{m}{2p}$.

So $v_2 = p\left(\frac{m}{2p}\right)^2 n = \frac{m^2 n}{4p}$.

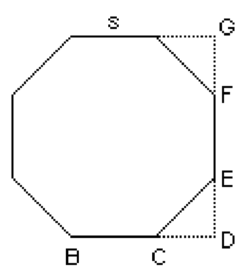
Then, $\frac{v_1}{v_2} = \frac{n}{m}$ and cylinder 2 has the greater volume.

16. *Describe elements which change and elements which do not change under transformations; ratio measures such as rates.* (H-2.3.3)
-10.6%

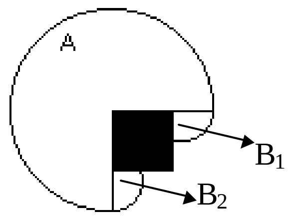
17. *How properties of geometric shapes relate to each other; use Pythagorean theorem.* H-2.2.4, H-2.2.2)

Extend sides $\overline{FE}$ and $\overline{BC}$ to D. ΔCDE is isosceles. So if $\overline{CE} = s$ and using the Pythagorean theorem, $\overline{CD} = \overline{ED} = \frac{1}{2}s\sqrt{2}$ and

$\overline{DG} = \overline{DE} + \overline{EF} + \overline{FG} = \frac{1}{2}s\sqrt{2} + s + \frac{1}{2}s\sqrt{2} = s + s\sqrt{2}$.



18. *How properties of geometric shapes relate to each other.* (H-2.3.1)
    Area of A $= \frac{3}{4}p\,(18)^2$
    Area of $B_1 = \frac{1}{4}p\ (8)^2$
    Area of $B_2 = \frac{1}{4}p\ (8)^2$
    Total area $\approx$ 863.6 sq. ft.



19. *How algebraic procedures and geometric concepts are related.* (H-2.1.3)
    ΔABC is *m Ð B + 3m Ð B + 3m Ð B = 180°*

    So, $m\angle B = \left(\frac{180}{7}\right)^{\circ}$

20. (H-2.1.4)
    Radius of smaller log is 2. In general, the radius of the smaller log is 1/4 the radius of the larger logs.